$3/4 + r(1/s\_a + 1/s\_b + 1/s\_c) \le s^2/(12r^2)$

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A triangle has perimeter $2s$, inradius r and the distances from its incentr to vertices are s_a, s_b, s_c. Prove that

$3/4 + r(1/s_a + 1/s_b + 1/s_c) \le s^2/(12r^2)$.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Noting that $\dfrac{r}{s_a} = \sin\dfrac{A}{2}, \dfrac{r}{s_b} = \sin\dfrac{B}{2}, \dfrac{r}{s_c} = \sin\dfrac{C}{2}$ we obtain

$3/4 + r(1/s_a + 1/s_b + 1/s_c) \le 3/4 + \sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2}$.

Since* $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} \le \dfrac{3}{2}$ and** $3\sqrt{3}\,r \le s \Leftrightarrow \dfrac{9}{4} \le \dfrac{s^2}{12r^2}$ we have

$3/4 + r(1/s_a + 1/s_b + 1/s_c) \le 3/4 + \sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} \le \dfrac{3}{4} + \dfrac{3}{2} = \dfrac{9}{4} \le \dfrac{s^2}{12r^2}$.

* $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} \le 3\sin\dfrac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3} = 3\sin\dfrac{\pi}{6} = \dfrac{3}{2}$(Jensen's Inequality for $\sin x$ because it is concave down on $[0, \pi]$).

Or, since $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = \cos\alpha + \cos\beta + \cos\gamma$, where $\alpha := \dfrac{\pi - A}{2}, \beta := \dfrac{\pi - B}{2}$, $\gamma := \dfrac{\pi - C}{2}$ and $\alpha, \beta, \gamma > 0,\ \alpha + \beta + \gamma = \pi$ then $\cos\alpha + \cos\beta + \cos\gamma = 1 + \dfrac{r_1}{R_1} \le \dfrac{3}{2}$ (because α, β, γ can be considered as angles of some triangle with inradius r_1 and circumradius R_1. And also we have $2r_1 \le R_1$ (Euler′s Inequality));

$\gamma := \dfrac{\pi - C}{2}$. Since $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi$ then

** By AM-GM Inequality

$r^2 s = (s-a)(s-b)(s-c) \le \left(\dfrac{s-a+s-b+s-c}{3}\right)^3 = \dfrac{s^3}{27} \Leftrightarrow 3\sqrt{3}\,r \le s$.